

COMPACT SUBSPACE OF PRODUCTS OF LINEARLY ORDERED SPACES AND CO-NAMIOKA SPACES

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ABSTRACT. It is shown that for any Baire space X , linearly ordered compact spaces Y_1, \dots, Y_n , compact space $Y \subseteq Y_1 \times \dots \times Y_n$ such that for every parallelepiped $W \subseteq Y_1 \times \dots \times Y_n$ the set $Y \cap W$ is connected, and separately continuous mapping $f : X \times Y \rightarrow \mathbb{R}$ there exists a dense in X G_δ -set $A \subseteq X$ such that f is jointly continuous at every point of $A \times Y$.

1. INTRODUCTION

Investigation of joint continuity points set of separately continuous function was started by R. Baire in his classical work [1] where he considered functions of two real variables. The Namioka's result [7] become the impulse to the intensification of these investigations. These result leads to appearance of the following notions which were introduced in [5].

Let X, Y be topological spaces. We say that a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ has the *Namioka property* if there exists a dense in X G_δ -set $A \subseteq X$ such that f is jointly continuous at every point of set $A \times Y$.

A compact space Y is called a *co-Namioka space* if for every Baire space X each separately continuous mapping $f : X \times Y \rightarrow \mathbb{R}$ has the Namioka property.

Very general results in the direction of study of co-Namioka space properties were obtained in [2, 3]. It was obtained in [2, 3] that the class of compact co-Namioka spaces is closed over products and contains all Valdivia compacts. Moreover, it was shown in [3] that the linearly ordered compact $[0, 1] \times \{0, 1\}$ with the lexicographical order is co-Namioka and it was reproved in [2] (result from [6]) that every completely ordered compact is co-Namioka. These results were generalized in [9]. It was shown in [9] that every linearly ordered compact is co-Namioka.

However, an example given by M.. Talagrand in [8] of a compact space which is not co-Namioka indicates that a closed subspace of co-Namioka compact can be not co-Namioka. Therefore the following question naturally arises: is every compact subspace Y of the product $Y_1 \times \dots \times Y_n$ of finite family of linearly ordered compacts Y_k a co-Namioka?

In this paper we using an approach from [9] we show that under some additional assumptions on Y the formulated above question has the positive answer.

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2. CONTINUOUS MAPPINGS ON COMPACT SUBSPACES OF PRODUCT LINEARLY ORDERED SPACES

For a mapping $f : X \times Y \rightarrow Z$ and a point $(x, y) \in X \times Y$ we put

$$f^x(y) = f_y(x) = f(x, y).$$

For a linearly ordered space X and points $a, b \in X$ with $a \leq b$ by $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) we denote the corresponding intervals.

Let X be a topological space. For a mapping $f : X \rightarrow \mathbb{R}$ and a set $A \subseteq X$ by $\omega_f(A)$ we denote the oscillation

$$\sup_{x, y \in A} |f(x) - f(y)|$$

of the function f on the set A . Moreover, for a point $x_0 \in X$ by $\omega_f(x_0)$ we denote the oscillation

$$\inf_{U \in \mathcal{U}} \omega_f(U)$$

of the function f at the point x_0 , where \mathcal{U} is a system of all neighborhoods of x_0 in X .

Proposition 2.1. *Let X_1, \dots, X_n be a linearly ordered spaces, $X \subseteq X_1 \times \dots \times X_n$ be a compact space, $\varepsilon > 0$ and $f : X \rightarrow \mathbb{R}$ be a continuous mapping. Then there exists an integer $m \in \mathbb{N}$ such that for every collection W_1, \dots, W_m of parallelepipeds*

$$W_k = [a_1^{(k)}, b_1^{(k)}] \times \dots \times [a_n^{(k)}, b_n^{(k)}]$$

such that $(a_i^{(k)}, b_i^{(k)}) \cap (a_i^{(j)}, b_i^{(j)}) = \emptyset$ for every $i \in \{1, \dots, n\}$ and distinct $j, k \in \{1, \dots, m\}$, there exists $k_0 \in \{1, \dots, m\}$ such that $\omega_f(W_{k_0} \cap X) \leq \varepsilon$.

Proof. Without loss of the generality we can propose that X_1, \dots, X_n are compacts. Let $g : X_1 \times \dots \times X_n \rightarrow \mathbb{R}$ be a continuous extension of the mapping f . We choose finite covers $\mathcal{U}_1, \dots, \mathcal{U}_n$ of spaces X_1, \dots, X_n by open intervals such that $\omega_g(\overline{U_1} \times \dots \times \overline{U_n}) \leq \varepsilon$ for every $U_1 \in \mathcal{U}_1, \dots, U_n \in \mathcal{U}_n$. For every $i \in \{1, \dots, n\}$ we denote by A_i the set of all ending points of intervals $U \in \mathcal{U}_i$ and put $m = |A_1| + |A_2| + \dots + |A_n| + 1$. Show that m is the required.

Let W_1, \dots, W_m be a collection of parallelepipeds

$$W_k = [a_1^{(k)}, b_1^{(k)}] \times \dots \times [a_n^{(k)}, b_n^{(k)}]$$

such that $(a_i^{(k)}, b_i^{(k)}) \cap (a_i^{(j)}, b_i^{(j)}) = \emptyset$ for every $i \in \{1, \dots, n\}$ and distinct $j, k \in \{1, \dots, m\}$. Suppose that $\omega_f(W_k \cap X) > \varepsilon$ for every $k \in \{1, \dots, m\}$. Then $W_k \not\subseteq \overline{U_1} \times \dots \times \overline{U_n}$ for every $k \in \{1, \dots, m\}$ and every $U_1 \in \mathcal{U}_1, \dots, U_n \in \mathcal{U}_n$. Therefore for every $k \in \{1, \dots, m\}$ there exist $i \in \{1, \dots, n\}$ and $a \in A_i$ such that $a \in (a_i^{(k)}, b_i^{(k)})$. Since $m > |A_1| + |A_2| + \dots + |A_n|$, there exist $i \in \{1, \dots, n\}$, $a \in A_i$ and distinct $j, k \in \{1, \dots, m\}$ such that $a \in (a_i^{(k)}, b_i^{(k)}) \cap (a_i^{(j)}, b_i^{(j)})$. But it is impossible. \square

3. MAIN RESULT

Theorem 3.1. *Let Y_1, \dots, Y_n be linearly ordered spaces, $Y \subseteq Y_1 \times \dots \times Y_n$ be a compact subspace such that for every (possibly empty) parallelepiped*

$$W = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq Y_1 \times \dots \times Y_n$$

the set $Y \cap W$ is connected. Then Y is co-Namioka.

Proof. Let X be a Baire space and $f : X \times Y \rightarrow \mathbb{R}$ be a separately continuous function. We prove that the mapping f has the Namioka property.

We fix $\varepsilon > 0$ and show that the open set

$$G_\varepsilon = \{x \in X : \omega_f(x, y) < \varepsilon \text{ for every } y \in Y\}$$

is dense in X .

Let U is an open in X nonempty set. Show that there exists an open nonempty set $U_0 \subseteq U \cap G_\varepsilon$. For every $x \in U$ we denote by $M(x)$ the set of all integers $m \in \mathbb{N}$ for which there exists collection W_1, \dots, W_m of parallelepipeds

$$W_k = [a_1^{(k)}, b_1^{(k)}] \times \dots \times [a_n^{(k)}, b_n^{(k)}]$$

such that $(a_i^{(k)}, b_i^{(k)}) \cap (a_i^{(j)}, b_i^{(j)}) = \emptyset$ for every $i \in \{1, \dots, n\}$ and distinct $j, k \in \{1, \dots, m\}$ and $\omega_{f^x}(W_k \cap Y) > \frac{\varepsilon}{6(n+1)}$ for every $k \in \{1, \dots, m\}$. According to Proposition 2.1 all sets $M(x)$ are upper bounded. For every $x \in U$ we put $\varphi(x) = \max M(x)$, if $M(x) \neq \emptyset$, and $\varphi(x) = 0$, if $M(x) = \emptyset$.

It follows from the continuity with respect to the first variable of the function f that for every parallelepiped $W \subseteq Y_1 \times \dots \times Y_n$ the set $\{x \in X : \omega_{f^x}(W \cap Y) > \frac{\varepsilon}{6(n+1)}\}$ is open in X . Therefore for every nonnegative $m \in \mathbb{Z}$ the set $\{x \in U : \varphi(x) > m\}$ is open in U , i.e. the function $\varphi : U \rightarrow \mathbb{Z}$ is lower semicontinuous on the Baire space U . According to [4] the function φ is pointwise discontinuous, i.e. φ is continuous at every point from some dense in U sets. There exist an open in U nonempty set U_1 and nonnegative real $m \in \mathbb{Z}$ such that $\varphi(x) = m$ for every $x \in U_1$.

If $m = 0$, then $|f(x, a) - f(x, b)| \leq \frac{\varepsilon}{6(n+1)} < \frac{\varepsilon}{3}$ for every $x \in U_1$ and $a, b \in Y$. Then taking a point $y_0 \in Y$ and an open nonempty set $U_0 \subseteq U_1$ such that $\omega_{f_{y_0}}(U_0) < \frac{\varepsilon}{3}$, we obtain that $\omega_f(U_0 \times Y) < \varepsilon$. In particular, $U_0 \subseteq G_\varepsilon$.

Now we consider the case of $m \in \mathbb{N}$. Take a point $x_0 \in U_1$ and choose a collection W_1, \dots, W_m of parallelepipeds

$$W_k = [a_1^{(k)}, b_1^{(k)}] \times \dots \times [a_n^{(k)}, b_n^{(k)}]$$

such that $(a_i^{(k)}, b_i^{(k)}) \cap (a_i^{(j)}, b_i^{(j)}) = \emptyset$ for every $i \in \{1, \dots, n\}$ and distinct $j, k \in \{1, \dots, m\}$ and $\omega_{f^{x_0}}(W_k \cap Y) > \frac{\varepsilon}{6(n+1)}$ for every $k \in \{1, \dots, m\}$. Using the continuity of f with respect to the first variable we choose an open neighborhood $U_0 \subseteq U_1$ of x_0 in U such that $\omega_{f^x}(W_k \cap Y) > \frac{\varepsilon}{6(n+1)}$ for every $k \in \{1, \dots, m\}$ and $x \in U_0$.

Further without loss of the generality we can propose that Y_1, \dots, Y_n are compacts, i.e. $Y_1 = [a_1, b_1], \dots, Y_n = [a_n, b_n]$. For every $i \in \{1, \dots, n\}$ we put

$$A_i = \{a_i^{(k)}, b_i^{(k)} : 1 \leq k \leq m\} \cup \{a_i, b_i\}$$

and denote by \mathcal{V}_i the set of all nonempty intervals $[a, b] \subseteq Y_i$ such that $a, b \in A_i$. Moreover, we put

$$\mathcal{W} = \mathcal{V}_1 \times \dots \times \mathcal{V}_n.$$

Show that $\omega_{f^x}(W \cap Y) \leq \frac{\varepsilon}{6}$ for all $x \in U_0$ $W \in \mathcal{W}$.

Suppose that $\omega_{f^x}(W \cap Y) > \frac{\varepsilon}{6}$ for some $x \in U_0$ and $W = [c_1, d_1] \times \dots \times [c_n, d_n] \in \mathcal{W}$. We choose $z_0 = (z_1^{(0)}, \dots, z_n^{(0)})$, $z_{n+1} = (z_1^{(n+1)}, \dots, z_n^{(n+1)}) \in W \cap Y$ such that $|f(x, z_0) - f(x, z_{n+1})| = q > \frac{\varepsilon}{6}$. For certainty we propose that $z_1^{(0)} \leq z_1^{(n+1)}, \dots, z_n^{(0)} \leq z_n^{(n+1)}$. Using the continuity of f^x and the fact that for every parallelepiped $V \subseteq Y_1 \times \dots \times Y_n$ the set $V \cap Y$ is connected, we choose points

$z_1 = (z_1^{(1)}, \dots, z_n^{(1)}), \dots, z_n = (z_1^{(n)}, \dots, z_n^{(n)}) \in W \cap Y$ such that $z_i^{(0)} \leq z_i^{(1)} \leq z_i^{(2)} \leq \dots \leq z_i^{(n)} \leq z_i^{(n+1)}$ for every $i \in \{1, \dots, n\}$ and $|f(x, z_{k-1}) - f(x, z_k)| = \frac{q}{n+1}$ for every $k \in \{1, \dots, n+1\}$. Now for every $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, n+1\}$ we put $c_i^{(k)} = z_i^{(k-1)}$, $d_i^{(k)} = z_i^{(k)}$ and $V_k = [c_1^{(k)}, d_1^{(k)}] \times \dots \times [c_n^{(k)}, d_n^{(k)}]$. Note that $(c_i^{(k)}, d_i^{(k)}) \cap (c_i^{(j)}, d_i^{(j)}) = \emptyset$ for every $i \in \{1, \dots, n\}$ and distinct $j, k \in \{1, \dots, n+1\}$ and $\omega_{f^x}(V_k \cap Y) \geq \frac{q}{n+1} > \frac{\varepsilon}{6(n+1)}$ for every $k \in \{1, \dots, n+1\}$.

Since for every $i \in \{1, \dots, n\}$ the set $\{k \leq m : (a_i^{(k)}, b_i^{(k)}) \cap (c_i, d_i) \neq \emptyset\}$ contains at most one element, for the set

$$N = \{k \leq m : (a_i^{(k)}, b_i^{(k)}) \cap (c_i, d_i) = \emptyset \forall i = 1, \dots, n\}$$

we have $|N| \geq m - n$. Therefore the system $\mathcal{P} = \{W_k : k \in N\} \cup \{V_j : 1 \leq j \leq n+1\}$ contains at least $m + 1$ parallelepipeds. Besides, $\omega_{f^x}(P \cap Y) > \frac{\varepsilon}{6(n+1)}$ for every $P \in \mathcal{P}$ and $(t_i, u_i) \cap (v_i, w_i) = \emptyset$ for every $i \in \{1, \dots, n\}$, where

$$[t_1, u_1] \times \dots \times [t_n, u_n], [v_1, w_1] \times \dots \times [v_n, w_n]$$

are distinct parallelepipeds with \mathcal{P} . But this contradicts to $\varphi(x) = m$. Thus, $\omega_{f^x}(W \cap Y) \leq \frac{\varepsilon}{6}$ for all $x \in U_0$ and $W \in \mathcal{W}$.

Fix $y \in Y$ and $x \in U_0$. Put $\mathcal{W}_y = \{W \in \mathcal{W} : y \in W\}$ and $V_y = \bigcup_{W \in \mathcal{W}_y} (W \cap Y)$.

Clearly that V_y is a neighborhood of y in Y . Since $\omega_{f^x}(W \cap Y) \leq \frac{\varepsilon}{6}$ for every $W \in \mathcal{W}_y$ and $y \in \bigcap_{W \in \mathcal{W}_y} (W \cap Y)$, $\omega_{f^x}(V_y) \leq \frac{\varepsilon}{3}$. Now using the continuity of f with

respect to the first variable at (x, y) we choose a neighborhood \tilde{U} of x in X such that $\omega_{f_y}(\tilde{U}) < \frac{\varepsilon}{3}$. Then $\omega_f(\tilde{U} \times V_y) < \varepsilon$, in particular, $\omega_f(x, y) < \varepsilon$ for every $y \in Y$ and $x \in U_0$. Thus, $U_0 \subseteq G_\varepsilon$. □

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